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Analytical Bethe ansatz for $A_{2n-1}^{(2)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ quantum-algebra-invariant open spin chains

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Abstract. We determine the eigenvalues of the transfer matrices for integrable open quantum spin chains which are associated with the affine Lie algebras $A_{2n-1}^{(2)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, and which have the quantum-algebra invariance $U_q(C_n)$, $U_q(B_n)$, $U_q(C_n)$, $U_q(D_n)$, respectively.

We recently generalized in [1] the analytical Bethe ansatz for integrable open spin chains with quantum-algebra invariance which was developed in [2] to the entire $A_{2n}^{(2)}$ series of $U_q(B_n)$-invariant spin chains in the fundamental representation. (The analytical Bethe ansatz for closed spin chains with periodic boundary conditions was formulated by Reshetikhin [3].) We focused in [1] on the $A_{2n}^{(2)}$ series because we sought to identify the main difficulties in generalizing the analytical Bethe ansatz procedure to any affine Lie algebra, and the $A_{2n}^{(2)}$ series was particularly convenient since the Izergin–Korepin [4] $A_2^{(2)}$ case was already understood [2]. The main difficulties were computing the pseudovacuum eigenvalue of the transfer matrix, and formulating an appropriate ansatz for general eigenvalues.

In [1] a 'doubling postulate' (i.e. that the Bethe ansatz equations are 'doubled' with respect to those of the corresponding closed chain with periodic boundary conditions) was introduced. This postulate was suggested by the structure of the Bethe ansatz equations in previously solved cases of open spin chains with quantum-algebra invariance. Using this 'doubling postulate', we were able to easily formulate an appropriate ansatz and obtain the transfer matrix eigenvalues‖. Very recently this procedure was used for the $G_2^{(1)}$ spin chain by Yung and Batchelor [5]. These authors have further generalized this method to certain open spin chains which are not quantum-algebra-invariant [5, 6].

The eigenvalues of the transfer matrix have been obtained (by means of the algebraic Bethe ansatz) also for the $A_n^{(1)}$ open spin chains [7, 8], and the $B_1^{(1)}$ open spin chain [9].

The success of the analytical Bethe ansatz procedure gives us confidence that the same procedure should work for the remaining series of quantum-algebra-invariant open spin chains, with Hamiltonian $\mathcal{H} = \sum_{j=1}^{N-1} d\check{R}_{j,j+1}(u)/du|_{u=0}$. Here $R(u)$ is the R matrix associated with an affine Lie algebra $g^{(k)}$ and with the fundamental representation of g_0, where $g_0 \subset g^{(k)}$ is the maximal finite-dimensional subalgebra of $g^{(k)}$. Unfortunately, the

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‖ This postulate provides a short-cut for obtaining the transfer matrix eigenvalues. In principle, this postulate can be avoided, and the transfer matrix eigenvalues obtained by carefully implementing the constraints of analyticity, crossing, fusion, asymptotic behaviour and periodicity. In practice, however, this can be quite tedious.

(diagonal) K matrix which is needed to construct the corresponding transfer matrix is known only for the following additional series of R matrices [10]: $A_{2n-1}^{(2)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$. We therefore restrict ourselves to these cases.

Specifically, in this paper we determine the eigenvalues of the transfer matrices for the following four infinite series of quantum-algebra invariant open spin chains:

- $U_q(C_n)$-invariant $A_{2n-1}^{(2)}$ spin chains;
- $U_q(B_n)$-invariant $B_n^{(1)}$ ($n > 1$) spin chains;
- $U_q(C_n)$-invariant $C_n^{(1)}$ spin chains;
- $U_q(D_n)$-invariant $D_n^{(1)}$ spin chains.

We emphasize that within the class of integrable open spin chains, those spin chains which are quantum-algebra invariant are in certain respects the *simplest* ones. Indeed, for the quantum-algebra-invariant spin chains, the Bethe ansatz states are highest-weight states of the quantum algebra [11–14]. Moreover, for those open spin chains which are not quantum-algebra invariant, the Bethe ansatz equations have additional factors depending on some boundary parameters.

We therefore work with solutions $R(u)$ [15, 16] of the Yang–Baxter equations which are associated with the affine Lie algebras $g^{(k)} = (A_{2n-1}^{(2)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)})$ and with the fundamental representation of the Lie algebras $g_0 = (C_n, B_n, C_n, D_n)$, respectively. The corresponding matrices $\check{R}(u)$ commute with the generators of the quantum algebras $U_q(g_0)$. In the appendix we collect the necessary information about these solutions. We follow the notation of [1].

Our goal is to determine the eigenvalues of the transfer matrix [7, 10]

$$t(u) = \text{tr}_a \, M_a \, T_a(u) \, \hat{T}_a(u) \tag{1}$$

where

$$
\begin{aligned}
T_a(u) &= R_{aN}(u) \, R_{aN-1}(u) \cdots R_{a1}(u) \\
\hat{T}_a(u) &= R_{1a}(u) \cdots R_{N-1a}(u) \, R_{Na}(u)
\end{aligned}
\tag{2}
$$

with the subscript a denoting the auxiliary space while the subscripts $1, \ldots, N$ refer to quantum spaces. The matrix M is given in the appendix. It is related to the crossing matrix V, $M = V^t V$, where

$$R_{12}(u) = V_1 \, R_{12}(-u - \rho)^{t_2} \, V_1 \tag{3}$$

with

$$
\rho = \begin{cases}
-\text{i}\pi - 2\kappa\eta & \text{for } A_{2n-1}^{(2)} \\
-2\kappa\eta & \text{for } B_n^{(1)}, \ C_n^{(1)}, \ D_n^{(1)}
\end{cases}
\tag{4}
$$

and $\kappa = (2n, \ 2n - 1, \ 2n + 2, \ 2n - 2)$ for $(A_{2n-1}^{(2)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)})$, respectively. The transfer matrix commutes with $U_q(g_0)$ [17, 12]. We consider simultaneous eigenstates of the transfer matrix $t(u)$ and the n Cartan generators $\{H_1, \ldots, H_n\}$ of $U_q(g_0)$. We call the corresponding eigenvalues $\Lambda^{(m_1, \ldots, m_n)}(u)$ and $\{\lambda_1, \ldots, \lambda_n\}$, respectively. The eigenvalues of the Cartan generators are related to the integers $m_1, \ldots, m_n$ by [3]†

$$\{\lambda_l\} = \{N - m_1, m_1 - m_2, \ldots, m_{n-1} - m_n\} \qquad \text{for } B_n^{(1)}$$

$$\{\lambda_l\} = \{N - m_1, m_1 - m_2, \ldots, m_{n-2} - m_{n-1} - m_n, m_{n-1} - m_n\} \qquad \text{for } D_n^{(1)}$$

$$\{\lambda_l\} = \{N - m_1, m_1 - m_2, \ldots, m_{n-1} - 2m_n\} \qquad \text{for } A_{2n-1}^{(2)}, C_n^{(1)}.$$

† Here we correct a typographical error in [3] for the case $C_n^{(1)}$.

We choose $\Lambda^{(m_1,...,m_n)}(u)$ to correspond to a highest-weight vector for the corresponding algebra $U_q(g_0)$.

To accomplish the analytical Bethe ansatz program we must specify the following additional information:

(i) The crossing relation [2]

$$\Lambda^{(m_1,...,m_n)}(u) = \Lambda^{(m_1,...,m_n)}(-u-\rho) \tag{5}$$

with ρ given by (4).

(ii) The fusion formula [2]

$$\tilde{\Lambda}^{(m_1,...,m_n)}(u) = \frac{1}{\alpha(u)^{2N}\,\beta(u)^2}\left\{\zeta(2u+2\rho)\,\Lambda^{(m_1,...,m_n)}(u)\,\Lambda^{(m_1,...,m_n)}(u+\rho)\right.$$

$$\left.-\zeta(u+\rho)^{2N}\,\mathfrak{g}(2u+\rho)\,\mathfrak{g}(-2u-3\rho)\right\}. \tag{6}$$

where

$$\zeta(u) = \mathfrak{g}(u)\mathfrak{g}(-u) \tag{7}$$

$$\mathfrak{g}(u) = \begin{cases} 2\sinh(\frac{u}{2}+2\eta)\cosh(\frac{u}{2}+\kappa\eta) & \text{for } A_{2n-1}^{(2)} \\ 2\sinh(\frac{u}{2}-2\eta)\sinh(\frac{u}{2}+\kappa\eta) & \text{for } B_n^{(1)},\ C_n^{(1)},\ D_n^{(1)} \end{cases} \tag{8}$$

and

$$\alpha(u) = \begin{cases} \cosh(\frac{u}{2}-\kappa\eta) & \text{for } A_{2n-1}^{(2)} \\ \sinh(\frac{u}{2}-\kappa\eta) & \text{for } B_n^{(1)},\ C_n^{(1)},\ D_n^{(1)} \end{cases} \tag{9}$$

$$\beta(u) = \sinh(u-2\kappa\eta).$$

The fusion formula will be used in order to check the correctness of the pseudovacuum eigenvalue.

(iii) The transfer matrix is periodic $t(u) = t(u+2\pi i)$, and its eigenvalues $\Lambda^{(m_1,...,m_n)}(u)$ are analytic functions of u.

To obtain the pseudovacuum eigenvalue of the transfer matrix we compute its expectation value in the pseudovacuum state for $N = 2, 3$ and we obtain

$$\Lambda^{(0,...,0)}(u) = c^{2N}\left\{e^{M_{11}\eta} - \frac{p_0\,\bar{e}^2}{b^2-c^2} + e^{M_{mm}\eta}\left[\frac{a_{m1}^2}{c^2-a_{mm}^2} + \frac{\bar{e}^2\,d^2}{(c^2-b^2)(c^2-a_{mm}^2)}\right]\right\}$$

$$+b^{2N}\left\{p_0(1+\frac{\bar{e}^2}{b^2-c^2}) + e^{M_{mm}\eta}\left[\frac{d^2}{b^2-a_{mm}^2} + \frac{\bar{e}^2\,d^2}{(b^2-c^2)(b^2-a_{mm}^2)}\right]\right\}$$

$$+a_{mm}^{2N}\,e^{M_{mm}\eta}\left\{1+\frac{d^2}{a_{mm}^2-b^2} + \frac{a_{m1}^2}{a_{mm}^2-c^2} + \frac{\bar{e}^2\,d^2}{(a_{mm}^2-c^2)(a_{mm}^2-b^2)}\right\} \tag{10}$$

where $p_0 = \sum_{i=2}^{m-1} M_{ii}$, with M_{ii} being the matrix elements of M given in (A19); $d^2 = \sum_{i=2}^{m-1} a_{mi}^2$; and $a_{\alpha\beta}$, b, c, and $\bar{e}$ are given in (A3), (A4) in the appendix. $m = 2n$ for $A_{2n-1}^{(2)}$, $C_n^{(1)}$, $D_n^{(1)}$ and $m = 2n+1$ for $B_n^{(1)}$.

We postulate that equation (10) is true for all N. Using Mathematica we find the following expressions for the pseudovacuum eigenvalue:

for $A^{(2)}_{2n-1}$:

$$\Lambda^{(0,...,0)}(u) = c(u)^{2N} \frac{\sinh(u - 2\kappa\eta)\cosh(u - \omega\eta)}{\sinh(u - 2\eta)\cosh(u - \kappa\eta)}$$

$$+ b(u)^{2N} p_0 \frac{\sinh(u)\sinh(u - 2\kappa\eta)}{\sinh(u - 2\eta)\sinh(u - 2(\kappa - 1)\eta)}$$

$$+ a_{mm}(u)^{2N} \frac{\sinh(u)\cosh(u - (2\kappa - \omega)\eta)}{\sinh(u - 2(\kappa - 1)\eta)\cosh(u - \kappa\eta)} \qquad (11)$$

for $B^{(1)}_n, C^{(1)}_n, D^{(1)}_n$:

$$\Lambda^{(0,...,0)}(u) = c(u)^{2N} \frac{\sinh(u - 2\kappa\eta)\sinh(u - \omega\eta)}{\sinh(u - 2\eta)\sinh(u - \kappa\eta)}$$

$$+ b(u)^{2N} p_0 \frac{\sinh(u)\sinh(u - 2\kappa\eta)}{\sinh(u - 2\eta)\sinh(u - 2(\kappa - 1)\eta)}$$

$$+ a_{mm}(u)^{2N} \frac{\sinh(u)\sinh(u - (2\kappa - \omega)\eta)}{\sinh(u - 2(\kappa - 1)\eta)\sinh(u - \kappa\eta)} \qquad (12)$$

where

$$p_0 = \begin{cases} \dfrac{2\sinh(\kappa\eta)\cosh((\kappa - 2)\eta)}{\sinh(2\eta)} & \text{for } B^{(1)}_n, D^{(1)}_n \\[3mm] \dfrac{2\sinh((2n - 2)\eta)\cosh(2n\eta)}{\sinh(2\eta)} & \text{for } A^{(2)}_{2n-1}, C^{(1)}_n \end{cases} \qquad (13)$$

$$\omega = \begin{cases} \kappa - 2 & \text{for } C^{(1)}_n \\[2mm] \kappa + 2 & \text{for } A^{(2)}_{2n-1}, B^{(1)}_n, D^{(1)}_n \end{cases} \qquad (14)$$

and $c(u)$, $b(u)$, $a_{mm}(u)$ are defined in the appendix. This eigenvalue is consistent with the fusion equation (6), i.e. the expression for $\tilde{\Lambda}^{(0,...,0)}(u)$ does not have poles for values of u for which $\alpha(u)$ or $\beta(u)$ are zero.

For the dressing of these eigenvalues we make the following ansatz.
For $A^{(2)}_{2n-1}$:

$$\Lambda^{(m_1,...,m_n)}(u) = A^{(m_1)}(u) \, c(u)^{2N} \frac{\sinh(u - 2\kappa\eta)\cosh(u - \omega\eta)}{\sinh(u - 2\eta)\cosh(u - \kappa\eta)}$$

$$+ C^{(m_1)}(u) a_{mm}(u)^{2N} \frac{\sinh(u)\cosh(u - (2\kappa - \omega)\eta)}{\sinh(u - 2(\kappa - 1)\eta)\cosh(u - \kappa\eta)}$$

$$+ b(u)^{2N} \left\{ \sum_{l=1}^{n-1} \left[z_l(u) \, B^{(m_l, m_{l+1})}_l(u) + \tilde{z}_l(u) \, \tilde{B}^{(m_l, m_{l+1})}_l(u) \right] \right\} \qquad (15)$$

for $B^{(1)}_n, C^{(1)}_n, D^{(1)}_n$:

$$\Lambda^{(m_1,...,m_n)}(u) = A^{(m_1)}(u) \, c(u)^{2N} \frac{\sinh(u - 2\kappa\eta)\sinh(u - \omega\eta)}{\sinh(u - 2\eta)\sinh(u - \kappa\eta)}$$

$$+ C^{(m_1)}(u) \, a_{mm}(u)^{2N} \frac{\sinh(u)\sinh(u - (2\kappa - \omega)\eta)}{\sinh(u - 2(\kappa - 1)\eta)\sinh(u - \kappa\eta)}$$

$$+ b(u)^{2N} \left\{ I\, w(u) \, B^{(m_n)}_n(u) + \sum_{l=1}^{n-1} \left[z_l(u) B^{(m_l, m_{l+1})}_l(u) + \tilde{z}_l(u) \, \tilde{B}^{(m_l, m_{l+1})}_l(u) \right] \right\}$$

$$\qquad (16)$$

where $I = 1$ for $B_n^{(1)}$ and $I = 0$ in the other cases. The functions A, B, C are the doubles of the corresponding expressions given by Reshetikhin [3] (apart from slight changes in notation) and so are also invariant under $u_j^{(l)} \to -u_j^{(l)}$:

$$A^{(m_1)}(u) = \prod_{j=1}^{m_1} \frac{\sinh(\frac{1}{2}(u - u_j^{(1)}) + \eta) \; \sinh(\frac{1}{2}(u + u_j^{(1)}) + \eta)}{\sinh(\frac{1}{2}(u - u_j^{(1)}) - \eta) \; \sinh(\frac{1}{2}(u + u_j^{(1)}) - \eta)} \qquad (17)$$

$$C^{(m_1)}(u) = A^{(m_1)}(-u - \rho)$$

$$B_l^{(m_l, m_{l+1})}(u) = \prod_{j=1}^{m_l} \frac{\sinh(\frac{1}{2}(u - u_j^{(l)}) - (l+2)\eta) \; \sinh(\frac{1}{2}(u + u_j^{(l)}) - (l+2)\eta)}{\sinh(\frac{1}{2}(u - u_j^{(l)}) - l\eta) \; \sinh(\frac{1}{2}(u + u_j^{(l)}) - l\eta)}$$

$$\times \prod_{j=1}^{m_{l+1}} \frac{\sinh(\frac{1}{2}(u - u_j^{(l+1)}) - (l-1)\eta) \; \sinh(\frac{1}{2}(u + u_j^{(l+1)}) - (l-1)\eta)}{\sinh(\frac{1}{2}(u - u_j^{(l+1)}) - (l+1)\eta) \; \sinh(\frac{1}{2}(u + u_j^{(l+1)}) - (l+1)\eta)}$$

$$\begin{cases} l = 1, \dots, n-2 & \text{for } A_{2n-1}^{(2)}, C_n^{(1)} \\ l = 1, \dots, n-1 & \text{for } B_n^{(1)} \\ l = 1, \dots, n-3 & \text{for } D_n^{(1)} \end{cases} \qquad (18)$$

$$\tilde{B}_l^{(m_l, m_{l+1})}(u) = B_l^{(m_l, m_{l+1})}(-u - \rho)$$

$$\tilde{z}_l(u) = z_l(-u - \rho) \qquad l = 1, \dots, n-1 \qquad (19)$$

for $A_{2n-1}^{(2)}$:

$$B_{n-1}^{(m_{n-1}, m_n)}(u) = \prod_{j=1}^{m_{n-1}} \frac{\sinh(\frac{1}{2}(u - u_j^{(n-1)}) - (n+1)\eta) \; \sinh(\frac{1}{2}(u + u_j^{(n-1)}) - (n+1)\eta)}{\sinh(\frac{1}{2}(u - u_j^{(n-1)}) - (n-1)\eta) \; \sinh(\frac{1}{2}(u + u_j^{(n-1)}) - (n-1)\eta)}$$

$$\times \prod_{j=1}^{m_n} \frac{\sinh(u - u_j^{(n)} - 2(n-2)\eta) \; \sinh(u + u_j^{(n)} - 2(n-2)\eta)}{\sinh(u - u_j^{(n)} - 2n\eta) \; \sinh(u + u_j^{(n)} - 2n\eta)} \qquad (20)$$

for $B_n^{(1)}$:

$$B_n^{(m_n)}(u) = \prod_{j=1}^{m_n} \frac{\sinh(\frac{1}{2}(u - u_j^{(n)}) - (n-2)\eta) \; \sinh(\frac{1}{2}(u + u_j^{(n)}) - (n-2)\eta)}{\sinh(\frac{1}{2}(u - u_j^{(n)}) - n\eta) \; \sinh(\frac{1}{2}(u + u_j^{(n)}) - n\eta)}$$

$$\times \frac{\sinh(\frac{1}{2}(u - u_j^{(n)}) - (n+1)\eta) \; \sinh(\frac{1}{2}(u + u_j^{(n)}) - (n+1)\eta)}{\sinh(\frac{1}{2}(u - u_j^{(n)}) - (n-1)\eta) \; \sinh(\frac{1}{2}(u + u_j^{(n)}) - (n-1)\eta)} \qquad (21)$$

for $C_n^{(1)}$:

$$B_{n-1}^{(m_{n-1}, m_n)}(u) = \prod_{j=1}^{m_{n-1}} \frac{\sinh(\frac{1}{2}(u - u_j^{(n-1)}) - (n+1)\eta) \; \sinh(\frac{1}{2}(u + u_j^{(n-1)}) - (n+1)\eta)}{\sinh(\frac{1}{2}(u - u_j^{(n-1)}) - (n-1)\eta) \; \sinh(\frac{1}{2}(u + u_j^{(n-1)}) - (n-1)\eta)}$$

$$\times \prod_{j=1}^{m_n} \frac{\sinh(\frac{1}{2}(u - u_j^{(n)}) - (n-3)\eta) \; \sinh(\frac{1}{2}(u + u_j^{(n)}) - (n-3)\eta)}{\sinh(\frac{1}{2}(u - u_j^{(n)}) - (n+1)\eta) \; \sinh(\frac{1}{2}(u + u_j^{(n)}) - (n+1)\eta)} \qquad (22)$$

for $D_n^{(1)}$:

$$B_{n-2}^{(m_{n-2},m_{n-1},m_n)}(u) = \prod_{j=1}^{m_{n-2}} \frac{\sinh(\frac{1}{2}(u-u_j^{(n-2)}) - n\eta) \ \sinh(\frac{1}{2}(u+u_j^{(n-2)}) - n\eta)}{\sinh(\frac{1}{2}(u-u_j^{(n-2)}) - (n-2)\eta) \ \sinh(\frac{1}{2}(u+u_j^{(n-2)}) - (n-2)\eta)}$$

$$\times \prod_{j=1}^{m_{n-1}} \frac{\sinh(\frac{1}{2}(u-u_j^{(n-1)}) - (n-3)\eta) \ \sinh(\frac{1}{2}(u+u_j^{(n-1)}) - (n-3)\eta)}{\sinh(\frac{1}{2}(u-u_j^{(n-1)}) - (n-1)\eta) \ \sinh(\frac{1}{2}(u+u_j^{(n-1)}) - (n-1)\eta)}$$

$$\times \prod_{j=1}^{m_n} \frac{\sinh(\frac{1}{2}(u-u_j^{(n)}) - (n-3)\eta) \ \sinh(\frac{1}{2}(u+u_j^{(n)}) - (n-3)\eta)}{\sinh(\frac{1}{2}(u-u_j^{(n)}) - (n-1)\eta) \ \sinh(\frac{1}{2}(u+u_j^{(n)}) - (n-1)\eta)}$$

$$B_{n-1}^{(m_{n-1},m_n)}(u) = \prod_{j=1}^{m_{n-1}} \frac{\sinh(\frac{1}{2}(u-u_j^{(n-1)}) - (n-3)\eta) \ \sinh(\frac{1}{2}(u+u_j^{(n-1)}) - (n-3)\eta)}{\sinh(\frac{1}{2}(u-u_j^{(n-1)}) - (n-1)\eta) \ \sinh(\frac{1}{2}(u+u_j^{(n-1)}) - (n-1)\eta)}$$

$$\times \prod_{j=1}^{m_n} \frac{\sinh(\frac{1}{2}(u-u_j^{(n)}) - (n+1)\eta) \ \sinh(\frac{1}{2}(u+u_j^{(n)}) - (n+1)\eta)}{\sinh(\frac{1}{2}(u-u_j^{(n)}) - (n-1)\eta) \ \sinh(\frac{1}{2}(u+u_j^{(n)}) - (n-1)\eta)}.$$

$$(23)$$

We note that in contrast to the analytical Bethe ansatz procedure for closed spin chains, equations (15), (16) also contain the unknown functions $z_l(u)$, $w(u)$. As mentioned in the introduction, these functions can be determined by the so called doubling postulate, i.e. we demand that the Bethe ansatz equations obtained from the cancellation of poles in A, B, C be 'doubled' with respect to those in Reshetikhin's paper [3]. The doubled Bethe ansatz equations are

$$\left[\frac{\sinh(\frac{1}{2}u_k^{(1)} - \eta)}{\sinh(\frac{1}{2}u_k^{(1)} + \eta)} \right]^{2N} = \prod_{j\neq k}^{m_1} \frac{\sinh(\frac{1}{2}(u_k^{(1)} - u_j^{(1)}) - 2\eta) \ \sinh(\frac{1}{2}(u_k^{(1)} + u_j^{(1)}) - 2\eta)}{\sinh(\frac{1}{2}(u_k^{(1)} - u_j^{(1)}) + 2\eta) \ \sinh(\frac{1}{2}(u_k^{(1)} + u_j^{(1)}) + 2\eta)}$$

$$\times \prod_{j=1}^{m_2} \frac{\sinh(\frac{1}{2}(u_k^{(1)} - u_j^{(2)}) + \eta) \ \sinh(\frac{1}{2}(u_k^{(1)} + u_j^{(2)}) + \eta)}{\sinh(\frac{1}{2}(u_k^{(1)} - u_j^{(2)}) - \eta) \ \sinh(\frac{1}{2}(u_k^{(1)} + u_j^{(2)}) - \eta)}$$

$$(24)$$

$$1 = \prod_{j=1}^{m_{l-1}} \frac{\sinh(\frac{1}{2}(u_k^{(l)} - u_j^{(l-1)}) + \eta) \ \sinh(\frac{1}{2}(u_k^{(l)} + u_j^{(l-1)}) + \eta)}{\sinh(\frac{1}{2}(u_k^{(l)} - u_j^{(l-1)}) - \eta) \ \sinh(\frac{1}{2}(u_k^{(l)} + u_j^{(l-1)}) - \eta)}$$

$$\times \prod_{j\neq k}^{m_l} \frac{\sinh(\frac{1}{2}(u_k^{(l)} - u_j^{(l)}) - 2\eta) \ \sinh(\frac{1}{2}(u_k^{(l)} + u_j^{(l)}) - 2\eta)}{\sinh(\frac{1}{2}(u_k^{(l)} - u_j^{(l)}) + 2\eta) \ \sinh(\frac{1}{2}(u_k^{(l)} + u_j^{(l)}) + 2\eta)}$$

$$\times \prod_{j=1}^{m_{l+1}} \frac{\sinh(\frac{1}{2}(u_k^{(l)} - u_j^{(l+1)}) + \eta) \ \sinh(\frac{1}{2}(u_k^{(l)} + u_j^{(l+1)}) + \eta)}{\sinh(\frac{1}{2}(u_k^{(l)} - u_j^{(l+1)}) - \eta) \ \sinh(\frac{1}{2}(u_k^{(l)} + u_j^{(l+1)}) - \eta)}$$

$$(25)$$

where

$$l = 1, \ldots, n-1 \qquad \text{for } B_n^{(1)}$$
$$l = 1, \ldots, n-2 \qquad \text{for } A_{2n-1}^{(2)} C_n^{(1)}$$
$$l = 1, \ldots, n-3 \qquad \text{for } D_n^{(1)}.$$

$$(26)$$

Moreover, the Bethe ansatz equations corresponding to values of $l = 1, 2, \ldots, n$ which are not included in (26) are as follows:

for $B_n^{(1)}$:

$$1 = \prod_{j=1}^{m_{n-1}} \frac{\sinh(\frac{1}{2}(u_k^{(n)} - u_j^{(n-1)}) + \eta) \; \sinh(\frac{1}{2}(u_k^{(n)} + u_j^{(n-1)}) + \eta)}{\sinh(\frac{1}{2}(u_k^{(n)} - u_j^{(n-1)}) - \eta) \; \sinh(\frac{1}{2}(u_k^{(n)} + u_j^{(n-1)}) - \eta)}$$

$$\times \prod_{j \neq k}^{m_n} \frac{\sinh(\frac{1}{2}(u_k^{(n)} - u_j^{(n)}) - \eta) \; \sinh(\frac{1}{2}(u_k^{(n)} + u_j^{(n)}) - \eta)}{\sinh(\frac{1}{2}(u_k^{(n)} - u_j^{(n)}) + \eta) \; \sinh(\frac{1}{2}(u_k^{(n)} + u_j^{(n)}) + \eta)} \tag{27}$$

for $A_{2n-1}^{(2)}$:

$$1 = \prod_{j=1}^{m_{n-2}} \frac{\sinh(\frac{1}{2}(u_k^{(n-1)} - u_j^{(n-2)}) + \eta) \; \sinh(\frac{1}{2}(u_k^{(n-1)} + u_j^{(n-2)}) + \eta)}{\sinh(\frac{1}{2}(u_k^{(n-1)} - u_j^{(n-2)}) - \eta) \; \sinh(\frac{1}{2}(u_k^{(n-1)} + u_j^{(n-2)}) - \eta)}$$

$$\times \prod_{j \neq k}^{m_{n-1}} \frac{\sinh(\frac{1}{2}(u_k^{(n-1)} - u_j^{(n-1)}) - 2\eta) \; \sinh(\frac{1}{2}(u_k^{(n-1)} + u_j^{(n-1)}) - 2\eta)}{\sinh(\frac{1}{2}(u_k^{(n-1)} - u_j^{(n-1)}) + 2\eta) \; \sinh(\frac{1}{2}(u_k^{(n-1)} + u_j^{(n-1)}) + 2\eta)}$$

$$\times \prod_{j=1}^{m_n} \frac{\sinh(u_k^{(n-1)} - u_j^{(n)} + 2\eta) \; \sinh(u_k^{(n-1)} + u_j^{(n)} + 2\eta)}{\sinh(u_k^{(n-1)} - u_j^{(n)} - 2\eta) \; \sinh(u_k^{(n-1)} + u_j^{(n)} - 2\eta)} \tag{28}$$

$$1 = \prod_{j=1}^{m_{n-1}} \frac{\sinh(u_k^{(n)} - u_j^{(n-1)} + 2\eta) \; \sinh(u_k^{(n)} + u_j^{(n-1)} + 2\eta)}{\sinh(u_k^{(n)} - u_j^{(n-1)} - 2\eta) \; \sinh(u_k^{(n)} + u_j^{(n-1)} - 2\eta)}$$

$$\times \prod_{j \neq k}^{m_n} \frac{\sinh(u_k^{(n)} - u_j^{(n)} - 4\eta) \; \sinh(u_k^{(n)} + u_j^{(n)} - 4\eta)}{\sinh(u_k^{(n)} - u_j^{(n)} + 4\eta) \; \sinh(u_k^{(n)} + u_j^{(n)} + 4\eta)} \tag{29}$$

for $C_n^{(1)}$:

$$1 = \prod_{j=1}^{m_{n-2}} \frac{\sinh(\frac{1}{2}(u_k^{(n-1)} - u_j^{(n-2)}) + \eta) \; \sinh(\frac{1}{2}(u_k^{(n-1)} + u_j^{(n-2)}) + \eta)}{\sinh(\frac{1}{2}(u_k^{(n-1)} - u_j^{(n-2)}) - \eta) \; \sinh(\frac{1}{2}(u_k^{(n-1)} + u_j^{(n-2)}) - \eta)}$$

$$\times \prod_{j \neq k}^{m_{n-1}} \frac{\sinh(\frac{1}{2}(u_k^{(n-1)} - u_j^{(n-1)}) - 2\eta) \; \sinh(\frac{1}{2}(u_k^{(n-1)} + u_j^{(n-1)}) - 2\eta)}{\sinh(\frac{1}{2}(u_k^{(n-1)} - u_j^{(n-1)}) + 2\eta) \; \sinh(\frac{1}{2}(u_k^{(n-1)} + u_j^{(n-1)}) + 2\eta)}$$

$$\times \prod_{j=1}^{m_n} \frac{\sinh(\frac{1}{2}(u_k^{(n-1)} - u_j^{(n)}) + 2\eta) \; \sinh(\frac{1}{2}(u_k^{(n-1)} + u_j^{(n)}) + 2\eta)}{\sinh(\frac{1}{2}(u_k^{(n-1)} - u_j^{(n)}) - 2\eta) \; \sinh(\frac{1}{2}(u_k^{(n-1)} + u_j^{(n)}) - 2\eta)} \tag{30}$$

$$1 = \prod_{j=1}^{m_{n-1}} \frac{\sinh(\frac{1}{2}(u_k^{(n)} - u_j^{(n-1)}) + 2\eta) \; \sinh(\frac{1}{2}(u_k^{(n)} + u_j^{(n-1)}) + 2\eta)}{\sinh(\frac{1}{2}(u_k^{(n)} - u_j^{(n-1)}) - 2\eta) \; \sinh(\frac{1}{2}(u_k^{(n)} + u_j^{(n-1)}) - 2\eta)}$$

$$\times \prod_{j \neq k}^{m_n} \frac{\sinh(\frac{1}{2}(u_k^{(n)} - u_j^{(n)}) - 4\eta) \; \sinh(\frac{1}{2}(u_k^{(n)} + u_j^{(n)}) - 4\eta)}{\sinh(\frac{1}{2}(u_k^{(n)} - u_j^{(n)}) + 4\eta) \; \sinh(\frac{1}{2}(u_k^{(n)} + u_j^{(n)}) + 4\eta)} \tag{31}$$

for $D_n^{(1)}$:

$$1 = \prod_{j=1}^{m_{n-3}} \frac{\sinh(\frac{1}{2}(u_k^{(n-2)} - u_j^{(n-3)}) + \eta) \; \sinh(\frac{1}{2}(u_k^{(n-2)} + u_j^{(n-3)}) + \eta)}{\sinh(\frac{1}{2}(u_k^{(n-2)} - u_j^{(n-3)}) - \eta) \; \sinh(\frac{1}{2}(u_k^{(n-2)} + u_j^{(n-3)}) - \eta)}$$

$$\times \prod_{j \neq k}^{m_{n-2}} \frac{\sinh(\frac{1}{2}(u_k^{(n-2)} - u_j^{(n-2)}) - 2\eta) \; \sinh(\frac{1}{2}(u_k^{(n-2)} + u_j^{(n-2)}) - 2\eta)}{\sinh(\frac{1}{2}(u_k^{(n-2)} - u_j^{(n-2)}) + 2\eta) \; \sinh(\frac{1}{2}(u_k^{(n-2)} + u_j^{(n-2)}) + 2\eta)}$$

$$\times \prod_{j=1}^{m_{n-1}} \frac{\sinh(\frac{1}{2}(u_k^{(n-1)} - u_j^{(n-1)}) + \eta) \; \sinh(\frac{1}{2}(u_k^{(n-1)} + u_j^{(n-1)}) + \eta)}{\sinh(\frac{1}{2}(u_k^{(n-1)} - u_j^{(n-1)}) - \eta) \; \sinh(\frac{1}{2}(u_k^{(n-1)} + u_j^{(n-1)}) - \eta)}$$

$$\times \prod_{j=1}^{m_n} \frac{\sinh(\frac{1}{2}(u_k^{(n-2)} - u_j^{(n)}) + \eta) \; \sinh(\frac{1}{2}(u_k^{(n-2)} + u_j^{(n)}) + \eta)}{\sinh(\frac{1}{2}(u_k^{(n-2)} - u_j^{(n)}) - \eta) \; \sinh(\frac{1}{2}(u_k^{(n-2)} + u_j^{(n)}) - \eta)} \tag{32}$$

$$1 = \prod_{j=1}^{m_{n-2}} \frac{\sinh(\frac{1}{2}(u_k^{(n-1)} - u_j^{(n-2)}) + \eta) \; \sinh(\frac{1}{2}(u_k^{(n-1)} + u_j^{(n-2)}) + \eta)}{\sinh(\frac{1}{2}(u_k^{(n-1)} - u_j^{(n-2)}) - \eta) \; \sinh(\frac{1}{2}(u_k^{(n-1)} + u_j^{(n-2)}) - \eta)}$$

$$\times \prod_{j \neq k}^{m_{n-1}} \frac{\sinh(\frac{1}{2}(u_k^{(n-1)} - u_j^{(n-1)}) - 2\eta) \; \sinh(\frac{1}{2}(u_k^{(n-1)} + u_j^{(n-1)}) - 2\eta)}{\sinh(\frac{1}{2}(u_k^{(n-1)} - u_j^{(n-1)}) + 2\eta) \; \sinh(\frac{1}{2}(u_k^{(n-1)} + u_j^{(n-1)}) + 2\eta)} \tag{33}$$

$$1 = \prod_{j=1}^{m_{n-2}} \frac{\sinh(\frac{1}{2}(u_k^{(n)} - u_j^{(n-2)}) + \eta) \; \sinh(\frac{1}{2}(u_k^{(n)} + u_j^{(n-2)}) + \eta)}{\sinh(\frac{1}{2}(u_k^{(n)} - u_j^{(n-2)}) - \eta) \; \sinh(\frac{1}{2}(u_k^{(n)} + u_j^{(n-2)}) - \eta)}$$

$$\times \prod_{j \neq k}^{m_n} \frac{\sinh(\frac{1}{2}(u_k^{(n)} - u_j^{(n)}) - 2\eta) \; \sinh(\frac{1}{2}(u_k^{(n)} + u_j^{(n)}) - 2\eta)}{\sinh(\frac{1}{2}(u_k^{(n)} - u_j^{(n)}) + 2\eta) \; \sinh(\frac{1}{2}(u_k^{(n)} + u_j^{(n)}) + 2\eta)} . \tag{34}$$

One can therefore determine the unknown functions $z_l(u)$:

$$z_l(u) = \frac{\sinh(u) \sinh(u - 2\kappa\eta) \cosh(u - \omega\eta)}{\sinh(u - 2l\eta) \sinh(u - 2(l+1)\eta) \cosh(u - \kappa\eta)}$$

$$l = 1, \ldots, n-1 \quad \text{for } A_{2n-1}^{(2)} \tag{35}$$

$$z_l(u) = \frac{\sinh(u) \sinh(u - 2\kappa\eta) \sinh(u - \omega\eta)}{\sinh(u - 2l\eta) \sinh(u - 2(l+1)\eta) \sinh(u - \kappa\eta)}$$

$$l = 1, \ldots, n-1 \quad \text{for } B_n^{(1)}, \; C_n^{(1)}, \; D_n^{(1)} \tag{36}$$

$$w(u) = \frac{\sinh(u) \sinh(u - 2\kappa\eta)}{\sinh(u - 2n\eta) \sinh(u - 2(n-1)\eta)} \quad \text{for } B_n^{(1)} \tag{37}$$

We have obtained expressions for the transfer matrix eigenvalues, equations (15)–(23), (35)–(37). These expressions pass a number of checks similar to those performed in [1] for the $A_{2n}^{(2)}$ case. We are therefore confident that these are the correct eigenvalues.

We conclude by listing some unsolved problems.

The cases which remain to be treated are $D_n^{(2)}$, $D_4^{(3)}$, and (with the exception of $G_2^{(1)}$) all the exceptional affine algebras. For these cases, the R and/or K matrices are not yet known.

As was noted in the introduction, the analytical Bethe ansatz method has been further generalized [5,6] to certain open spin chains which are not quantum-algebra-invariant; namely, spin chains for which the K matrix is diagonal but is not necessarily equal to the

identity matrix. (For a spin chain with a non-diagonal K matrix, the analytical Bethe ansatz method presumably does not work, since an eigenstate (e.g. the pseudovacuum state) of the transfer matrix is not available.) It would be interesting to find new diagonal K matrices, and to diagonalize the corresponding transfer matrices.

Other open problems include formulating the algebraic Bethe ansatz for open spin chains (this is known only for the cases $A_n^{(1)}$ and $B_1^{(1)}$); studying further examples of models with spins in higher-dimensional representations; and investigating 'graded' models associated with solutions of the graded Yang–Baxter equations. Perhaps the most interesting outstanding problem is to use the Bethe ansatz results to investigate boundary phenomena in the thermodynamic ($N \to \infty$) limit.

Acknowledgments

Appendix. The R matrix

The R matrices associated with the fundamental representation of $A_{2n-1}^{(2)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ were found by Bazhanov [15] and Jimbo [16]†. We follow the latter work; however, we use the variables u and η instead of x and k, respectively, which are related as follows:

$$x = e^u \qquad k = e^{2\eta}. \tag{A1}$$

The R matrix is given by‡

$$R(u) = c(u) \sum_{\alpha \neq \alpha'} E_{\alpha\alpha} \otimes E_{\alpha\alpha} + b(u) \sum_{\alpha \neq \beta, \beta'} E_{\alpha\alpha} \otimes E_{\beta\beta}$$

$$+ \left(e(u) \sum_{\alpha < \beta, \alpha \neq \beta'} + \bar{e}(u) \sum_{\alpha > \beta, \alpha \neq \beta'} \right) E_{\alpha\beta} \otimes E_{\beta\alpha} + \sum_{\alpha, \beta} a_{\alpha\beta}(u) E_{\alpha\beta} \otimes E_{\alpha'\beta'} \tag{A2}$$

with

$$\left. \begin{array}{l} c(u) = 2\sinh(\frac{u}{2} - 2\eta) \\ b(u) = 2\sinh(\frac{u}{2}) \\ e(u) = -2e^{-u/2}\sinh(2\eta) \\ \bar{e}(u) = e^u e(u) \end{array} \right\} \times \left\{ \begin{array}{ll} \cosh(\frac{u}{2} - \kappa\eta) & \text{for } A_{2n-1}^{(2)} \\ \sinh(\frac{u}{2} - \kappa\eta) & \text{for } B_n^{(1)}, \ C_n^{(1)}, \ D_n^{(1)} \end{array} \right. \tag{A3}$$

---

† The $A_{2n-1}^{(2)}$ R matrix given in [15, 16] is $U_q(D_n)$ invariant. We consider here a *different* $A_{2n-1}^{(2)}$ R matrix, which instead is $U_q(C_n)$ invariant. We obtain [18] the latter R matrix from the $C_n^{(1)}$ R matrix by replacing (in the notation of the first paper in [16]) $\xi = k^{2n+2}$ by $\xi = -k^{2n}$; i.e. by changing $\xi \to -\xi k^{-2}$. The R matrix obtained in this way presumably coincides with the $U_q(C_n)$-invariant $A_{2n-1}^{(2)}$ R matrix of Kuniba [19]. We remark that the $A_{2n-1}^{(2)}$ R matrix given in [16] can be obtained from that for $D_n^{(1)}$ by changing $\xi \to -\xi k^2$. Similarly, the $A_{2n}^{(2)}$ R matrix, which is $U_q(B_n)$ invariant, can be obtained from the $B_n^{(1)}$ R matrix by changing $\xi \to -\xi k^2$.

‡ This expression for the R matrix differs from the one given in [16] by the overall factor $2e^{u+(\kappa+2)\eta}$.

$$a_{\alpha\beta}(u) = \begin{cases} 2\sinh(\frac{u}{2}) \times \begin{cases} \cosh(\frac{u}{2} - (\kappa-2)\eta) & \text{for } A^{(2)}_{2n-1} \\ \sinh(\frac{u}{2} - (\kappa-2)\eta) & \text{for } B^{(1)}_n, C^{(1)}_n, D^{(1)}_n \end{cases} & \alpha = \beta, \alpha \neq \alpha' \\[2em] b(u) + 2\sinh(2\eta)\sinh(2n-1)\eta \quad \text{for } B^{(1)}_n \quad \alpha = \beta, \alpha = \alpha' \\[1em] 2\sinh 2\eta\, e^{\mp u/2} \times \begin{cases} \mp\epsilon_\alpha\epsilon_\beta e^{(\pm\kappa+2(\bar\alpha-\bar\beta))\eta}\sinh\frac{u}{2} \\ \quad -\delta_{\alpha\beta'}\cosh(\frac{u}{2}-\kappa\eta) & \text{for } A^{(2)}_{2n-1} \\ \epsilon_\alpha\epsilon_\beta e^{(\pm\kappa+2(\bar\alpha-\bar\beta))\eta}\sinh\frac{u}{2} \\ \quad -\delta_{\alpha\beta'}\sinh(\frac{u}{2}-\kappa\eta) & \text{for } B^{(1)}_n, C^{(1)}_n, D^{(1)}_n \end{cases} & \alpha \lessgtr \beta \end{cases}$$

$$\tag{A4}$$

where

$$\kappa = \begin{cases} 2n & \text{for } A^{(2)}_{2n-1} \\ 2n-1 & \text{for } B^{(1)}_n \\ 2n+2 & \text{for } C^{(1)}_n \\ 2n-2 & \text{for } D^{(1)}_n \end{cases} \tag{A5}$$

and

$$\bar\alpha = \begin{cases} \alpha - \frac{1}{2} & 1 \leqslant \alpha \leqslant n \\ \alpha + \frac{1}{2} & n+1 \leqslant \alpha \leqslant 2n \end{cases} \quad \text{for } A^{(2)}_{2n-1}, C^{(1)}_n \tag{A6}$$

$$\bar\alpha = \begin{cases} \alpha + \frac{1}{2} & 1 \leqslant \alpha < \dfrac{\tilde N + 1}{2} \\ \alpha & \alpha = \dfrac{\tilde N + 1}{2} \\ \alpha - \frac{1}{2} & \dfrac{\tilde N + 1}{2} < \alpha \leqslant \tilde N \end{cases} \quad \text{for } B^{(1)}_n, D^{(1)}_n \tag{A7}$$

$$\begin{aligned} &\alpha, \beta = 1, \ldots, \tilde N \\ &\alpha' = \tilde N + 1 - \alpha \end{aligned} \tag{A8}$$

$$\begin{aligned} \epsilon_\alpha &= \begin{cases} 1 & \text{for } 1 \leqslant \alpha \leqslant n \\ -1 & \text{for } n+1 \leqslant \alpha \leqslant 2n \end{cases} \quad \text{for } A^{(2)}_{2n-1}, C^{(1)}_n \\ \epsilon_\alpha &= 1 \quad \text{for } B^{(1)}_n, D^{(1)}_n \end{aligned} \tag{A9}$$

where $\tilde N = 2n$ for $A^{(2)}_{2n-1}$, $C^{(1)}_n$, $D^{(1)}_n$ and $\tilde N = 2n+1$ for $B^{(1)}_n$; and the $E_{\alpha\beta}$ are elementary matrices. Evidently, the R matrix acts on the tensor product space $C^{\tilde N} \otimes C^{\tilde N}$.

In addition to obeying the Yang–Baxter equation, this R matrix satisfies the following important properties.

PT symmetry.

$$\mathcal{P}_{12}\, R_{12}(u)\, \mathcal{P}_{12} \equiv R_{21}(u) = R_{12}(u)^{t_1 t_2}. \tag{A10}$$

Unitarity.

$$R_{12}(u)\ R_{21}(-u) = \zeta(u) \tag{A11}$$

where $\zeta(u)$ is given by

$$\zeta(u) = \begin{cases} -4\sinh(\frac{u}{2} - 2\eta)\cosh(\frac{u}{2} - \kappa\eta) \\ \qquad \times \sinh(\frac{u}{2} + 2\eta)\cosh(\frac{u}{2} + \kappa\eta) & \text{for } A_{2n-1}^{(2)} \\ 4\sinh(\frac{u}{2} - 2\eta)\sinh(\frac{u}{2} - \kappa\eta) \\ \qquad \times \sinh(\frac{u}{2} + 2\eta)\sinh(\frac{u}{2} + \kappa\eta) & \text{for } B_n^{(1)},\ C_n^{(1)},\ D_n^{(1)}. \end{cases} \tag{A12}$$

Crossing symmetry.

$$R_{12}(u) = V_1\ R_{12}(-u-\rho)^{t_2}\ V_1 = V_2^{t_2}\ R_{12}(-u-\rho)^{t_1}\ V_2^{t_2} \tag{A13}$$

where $\rho = -\mathrm{i}\pi - 2\kappa\eta$ for $A_{2n-1}^{(2)}$ and $\rho = -2\kappa\eta$ for $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$; also $V^2 = 1$.

Regularity.

$$R(0) = -\zeta(0)^{\frac{1}{2}}\mathcal{P} \tag{A14}$$

where $\mathcal{P}$ is the permutation operator

$$\mathcal{P} = \sum_{\alpha,\beta} E_{\alpha\beta} \otimes E_{\beta\alpha}. \tag{A15}$$

Commutativity.

$$\left[\check{R}(u), \check{R}(v)\right] = 0 \qquad \check{R} = \mathcal{P}\mathcal{R}. \tag{A16}$$

Periodicity.

$$R(u + 2\pi\mathrm{i}) = R(u). \tag{A17}$$

The crossing matrix is given by

$$V = E_{\alpha\alpha}\delta_{\alpha\alpha'} - \sum_{\alpha\neq\alpha'} \mathrm{e}^{(\bar{\alpha}-\bar{\alpha}')\eta} E_{\alpha\alpha'}. \tag{A18}$$

Correspondingly, $M = V^t\ V$ is given by the $\tilde{N} \times \tilde{N}$ diagonal matrix

$$M = $$
$$\begin{cases} \mathrm{diag}\left(\mathrm{e}^{2(2n)\eta}, \mathrm{e}^{2(2n-2)\eta}, \ldots, \mathrm{e}^{4\eta}, \mathrm{e}^{-4\eta}, \ldots, \mathrm{e}^{-2(2n-2)\eta}, \mathrm{e}^{-2(2n)\eta}\right) & \text{for } A_{2n-1}^{(2)}, C_n^{(1)} \\ \mathrm{diag}\left(\mathrm{e}^{2(2n-1)\eta}, \mathrm{e}^{2(2n-3)\eta}, \ldots, \mathrm{e}^{2\eta}, 1, \mathrm{e}^{-2\eta}, \ldots, \mathrm{e}^{-2(2n-3)\eta}, \mathrm{e}^{-2(2n-1)\eta}\right) & \text{for } B_n^{(1)} \\ \mathrm{diag}\left(\mathrm{e}^{2(2n-2)\eta}, \mathrm{e}^{2(2n-4)\eta}, \ldots, 1, 1, \ldots, \mathrm{e}^{-2(2n-4)\eta}, \mathrm{e}^{-2(2n-2)\eta}\right) & \text{for } D_n^{(1)}. \end{cases}$$
$$\tag{A19}$$

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